# Numbers and data

This week we'll look at how the system of numbers we use has arisen, and review some basic algebra. While we can always do these calculations using software, the better we understand how to do them ourselves, the easier it is to focus on *how* those calculations can help us "do science". Numbers in science usually come from measurements, which are accompanied by *units* and *errors*, which we will look at this week. The results of these measurements can vary over a huge scale, from the subatomic to the galactic, so we will also consider convenient ways of representing numbers over such a range. Most of the ideas in this week's notes should be familiar to you, and are fundamental to QR.

# Where do numbers come from?

Humans first used numbers to count the things around them. Soon, we realised that certain rules always held, *independent of what it was that was being counted*. If I combine two apples and three apples, I get five apples; but the same holds if I'm counting apples, or sheep, or minutes until the end of class. We represent this in an abstracted form that removes any reference to the objects, and focussed purely on the numbers themselves:

$$2 + 3 = 5$$

A different type of rule tells me that two groups of three apples contain six apples. Again, this rule is independent of the object being counted, and we write it in a way that drops any reference to the object involved, referring only to the numbers:

$$2 \text{ x } 3 = 6$$

We learn these rules in school, and learn to add and multiply more complicated numbers, but the starting point of all this is an application of the scientific method to processes involving counting, turning observation of the world around us into rules.

These numbers 1, 2, 3, 4, … are the *natural numbers* (symbol $\mathbb{N}$).

## Algebra and the Natural Numbers

*Algebra* is the study of the operations of addition, multiplication and their inverses (subtraction and division) on sets of numbers. This also includes multiplying many times, which is represented by raising to a power, as in

- $2 \times 2 \times 2 = 2^3 = 8$
- $10 \times 10 = 10^2 = 100$

We notice certain rules that hold for the natural numbers:

- Commutativity for + and x: $a + b = b + a$ and $a \times b = b \times a$
- Associativity for + and x: $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$
- Distributivity: $a \times (b + c) = (a \times b) + (a \times c)$

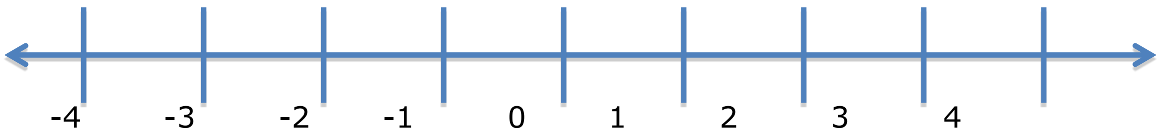
## Beyond the natural numbers

The natural numbers are *closed* under addition and multiplication: adding any two natural numbers always produces another natural number, as does multiplying any two natural numbers.

However, this isn't the case with *subtracting*. For example, 3 – 5 has no answer among the natural numbers. For counting objects, that seems fair enough, but for tracking debts, or values relative to some arbitrary zero (eg height above sea level), asking *'what is 5 – 3?'* is a reasonable question. To answer it, we invent the *integers* (symbol $\mathbb{Z}$), which includes 0 and all the negative whole numbers, as well as all the natural numbers.

We encounter the same problem with division. 5/3 has no answer among the natural numbers (or the integers). Again, if I have I can chop into smaller parts, like apples, it makes sense to ask how I can divide 5 apples among 3 people, ie *'what is 5/3?'*. To answer it, we invent the *rational numbers* (symbol $\mathbb{Q}$), which includes numbers that are fractions of two integers. It therefore includes all the integers as well, as any integer can be written as a fraction with denominator 1 (5=5/1)

We can represent all these numbers as points on a *number line*, which we can think of as the set of all possible measurements (lengths, for example). However, there are still some lengths that can't be written as a fraction of two integers – numbers like $\pi$, or e, or $\sqrt{2}$. So the number line contains all the rational numbers, but many others as well. We call the set of all possible numbers on the number line the *real numbers* (symbol $\mathbb{R}$).[1]




A number line with the integers marked. The number line represents all possible measurable numbers, ie the *real numbers*

## Magnitudes (absolute values)

The magnitude of a number $x$, denoted $|x|$, is its distance from 0. It is always positive.

For example: |-1| is the distance between -1 and 0, which is 1.

Practice: what are |4|, |-3|, |-1.5|, $|-\pi|$ ?

## Greater than and less than

*"Greater than"* means "*in the direction of increasing numbers from*"

*"Less than"* means "*in the direction of decreasing numbers from*"

"Greater than" and "less than" *have nothing to do with magnitude*. They only involve relative positions on the number line.

For example: -6 < 3, since -6 is in the direction of decreasing numbers from 3.

Practice: Fill in the missing signs (< or >)

7 __ 2 7 __ -2 -7 __ -2 -7 __ 2

[1] This process continues if we ask "*what is the set of all possible solutions to real polynomial equations?*" . The answer is the set of *complex numbers* (symbol )$\mathbb{C}$, but these are beyond the scope of this course.

# Quick review of rules of algebra with negative numbers and fractions

## Rules for adding and subtracting ***positive*** numbers (using number line)

To add: move in the positive direction (e.g. 3 + 5 =      )

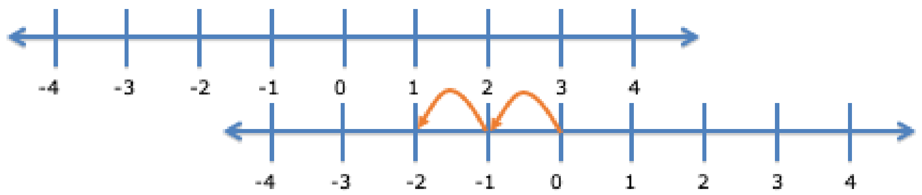
To subtract: moving in the negative direction (e.g. 3 – 5 =      )

## Rules for adding and subtracting ***negative*** numbers (using number line)

a + (-b) = a – b     and     a – (-b) = a + b

We can think of adding negative numbers as an extension to adding positive numbers in the following way. To add 3+(-2)

- place a new number line centred at 3, and hop to (-2) on the new number line




- move back to the original number line at that point: we see that 3 + (-2) = 1
- this this is the same action as subtracting 2 from 3

Subtracting then becomes the opposite action (hopping the opposite way to adding).

Practice examples: what are 4 + (-5), 5 – (-4), -6 + (-3), -3 + 4 ?

Rules for multiplying and dividing positive and negative numbers

- Multiplying/dividing with an odd number of negatives gives a *negative* number
- Multiplying/dividing with an even number of negatives gives a *positive* number

For example,

$$3 \times 5 = 15, \quad \text{so}$$
$$3 \times (-5) = -15$$
$$(-3) \times 5 = -15$$
$$(-3) \times (-5) = 15$$

and

$$\frac{4}{8} = \frac{1}{2}, \quad so$$

$$\frac{-4}{8} = -\frac{1}{2}$$

$$\frac{4}{-8} = -\frac{1}{2}$$

$$\frac{-4}{-8} = \frac{1}{2}$$

This rule extends for multiplying and/or dividing more than just two numbers, eg

$(-1) \times (-2) \times (-3) \times (-4) = (1 \times 2 \times 3 \times 4) = 24$, as there are an even number of negatives, whereas $(-1) \times (-2) \times 3 \times (-4) = -(1 \times 2 \times 3 \times 4) = -24$, as there are an odd number of negatives.

Similarly, $\frac{(-2)\times(-3)\times 6}{4\times(-9)} = \frac{36}{-36} = -1$. Notice the odd number of negative numbers.

Practice examples: what are $(-4) \times 6$, $(-2) \times (-1) \times (-3)$, $\frac{2}{(-4)}$, $\frac{2\times(-3)}{(-4)\times(-5)}$ ?

## Rules for adding and subtracting fractions

To *add* or *subtract* fractions, you must convert them to a common denominator

Examples:

$$\frac{3}{4}+\frac{1}{5}=$$

$$\frac{2}{3}-\frac{1}{5}=$$

$$2-\frac{1}{3}=$$

Notice that whole numbers can be written as fractions over 1, i.e.

Practice problems:

$$\frac{1}{3}+\frac{2}{5}=$$

$$\frac{1}{3}-\frac{2}{5}=$$

## Rules for multiplying and dividing fractions

To multiply fractions, we multiply numerators and denominators

To divide fractions, we invert the second (dividing) fraction and multiply

Examples:

$$\frac{3}{4}\times\frac{2}{5}$$

$$\frac{3}{4}\div\frac{2}{5}=$$

$$\frac{2}{3}\div 3=$$

Practice:

$\frac{2}{3} \times \frac{6}{5} =$

$\frac{7}{3} \div 4 =$

## Rules for exponents

$a^0 = 1$ for any number except 0 $(0^0 = 0)$ $a^1 = a$

$a^m \times a^n = a^{(m+n)}$ $\frac{a^m}{a^n} = a^{(m-n)}$ $a^{(-n)} = \frac{1}{a^n}$ $a^{1/n} = \sqrt[n]{a}$

For example:

$2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$

$\frac{2^5}{2^3} = \frac{2\times2\times2\times2\times2}{2\times2\times2} = \frac{2\times2}{1} = 2^2$ and $\frac{2^3}{2^5} = \frac{2\times2\times2}{2\times2\times2\times2\times2} = \frac{1}{2\times2} = 2^{-2}$

$\sqrt{2} = 2^{1/2}$ because $\sqrt{2} \times \sqrt{2} = 2^{1/2} \times 2^{1/2} = 2^{1/2+1/2} = 2$

Practice:

$3^3 =$

$\frac{3^7}{3^5} =$

$\frac{3^4}{3^6} =$

## Representing fractions as percentages

A percentage is the numerator of a fraction out of 100

Examples:

$$\frac{2}{5} = \frac{40}{100} = 40\% \qquad \frac{1}{3} \approx \frac{33.3}{100} = 33.3\% \qquad 2 = \frac{200}{100} = 200\%$$

Practice: What are 4/5 and 5/4, expressed as percentages?

## Order of operations: BODMAS

Operations are done in the following order:

- Brackets
- 'Order' (exponents)
- Division and Multiplication (left to right)
- Addition and Subtraction (left to right)

Examples:

$7 \text{ x } (3 - 2^2 + 5) =$

$7 \text{ x } 7 + 7 - 7 + 7 \div 7 =$

$1+(2-(3-(4-5))) =$

# Units

Measuring a quantity results in a number on an agreed scale. That scale reflects a choice of ***units*** (definite predetermined amounts): standardising the scale allows comparison between different amounts measured against that scale.

In Australia (like in Europe), we use Système International (SI) units:

- Lengths in metres (m)
- Mass in kilograms (kg)
- Time in seconds (s)
- Temperature in kelvin (K)
- Electrical current in amperes (A)
- Amount of substance in moles (mol)
- Luminous intensity in candelas (cd)

*Everything else* can be written as some combination of these, although common combinations are given their own name.

- *speed = distance/time*, so its units are m/s (or $ms^{-1}$)
- *pressure* has units of $kgm^{-1}s^{-2}$, otherwise called *pascals*
- Other better-known examples: joules, watts, volts, hertz, newtons

Examples:

- Acceleration = rate of change of velocity/time, so units are …
- Power = rate of change of energy/time, so units are …
- Energy = ½ x mass x velocity$^2$, so units are …
- Energy = mass x acceleration due to gravity x height, so units are …
- Diffusivity = distance$^2$ / time, so units are …
- Units of force (=mass x acceleration)
- Units of frequency (=number per second)
- Units of radians (=arclength/radius)

## Non-SI units

There are a number of non-SI units accepted for use alongside SI units. Also, Americans use their own customary units (related to British Imperial).

Examples on non-SI units accepted for use alongside SI units:

Examples on US customary units:

## Converting units

Use the approach of cancelling units. For example, to convert the speed limit 60 km/h to m/s, we note that 1 km = 1000 m and 1 h = 60 min = 60 x 60 s = 3600 s. Therefore

$$60\ km/h\ = \frac{60\ km}{1\ hr} = \frac{60\ km}{1\ h} \times \frac{1000\ m}{1\ km} \times \frac{1\ h}{3600\ s} = \frac{60}{3.6}\frac{\cancel{km}\times m\times \cancel{h}}{\cancel{h}\times \cancel{km}\times s} = 16.7\ \ m/s$$

Practice changing units

- What is the world record for the women's 4x100m freestyle?
- How fast is that (average speed) in m/s?
- How fast is that (average speed) in km/h

Exercise: Finding relations between units

My mower can handle mowing a lawn at about $1m^2/s$. How many minutes would it take me to mow my parents' ¼-acre lawn?

## The Decimal System (click here for the YouTube clip)

We express numbers in terms of the 10 digits 0 to 9, using the *decimal system*. The placement of digits in relation to the decimal point (omitted at the end of whole numbers) gives them a specific meaning. Every real number can be expressed in this way.

For example, $302.14 = 3 \times 10^2 + 0 \times 10^1 + 2 \times 10^0 + 1 \times 10^{-1} + 4 \times 10^{-2}$

$= 300 + 2 + 0.1 + 0.04$

All rationals have either finite or repeating decimal expansions

- $1/10 = 0.1$
- $1/9 = 0.111\ldots = 0.\overline{1} = 1/10 + 1/100 + 1/1000 + \ldots$
- $1/8 = 0.125 = 1/10 + 2/100 + 5/1000$
- $1/7 = 0.142857142857\ldots = 0.\overline{142857} = 1/10 + 4/100 + 2/1000 + \ldots$ (=142857/999999)

---

*Science History Sidebar!*

The first applied implementation of the metric system came in 1799 during the French Revolution. It was adopted because the many systems of measures had become impractical for trade. It was set up initially as a decimal system based on the kilogram and the metre.

---

## Scientific notation

The range of values of measured scientific quantities is *vast*. For example, the radius of the smallest atomic nucleus is about 0.000000000000001 m, while the size of our galaxy is about 1000000000000000000000 m. We need a way of conveniently expressing these values.

Recall that $10^2$=10x10=100, $10^3$=10x10x10=1000, $10^4$=10x10x10x10=10000, etc., so

- $10^n$=10x10x…x10 – a `1' with n `0' after it

So we can express huge numbers like Avogadro's number *N*=602214076000000000000000, the number of molecules in 12 g of Carbon-12, in a more convenient way:

$$N = 6.02214076 \times 100,000,000,000,000,000,000,000 = 6.02214076 \times 10^{23}$$

To express tiny numbers like the separation of carbon atoms on a graphene sheet (0.00000000142 m), note that $10^{-1}$=1/10=0.1, $10^{-2}$=1/(10x10)=0.01, and $10^{-3}$=1/(10x10x10)=0.001, etc., so we can write

$$\text{separation is } 0.000000000142 \text{ m} = 1.42 \times 0.0000000001 = 1.42 \times 10^{-10} \text{ m}$$

We have written these numbers in *normalized scientific notation*, that is, in the form

$$\pm r \times 10^m$$

where $r$ is a real number (called the *significand*) with $1 \leq r < 10$ ($r$ can equal 1, but not 10); and $m$ is an integer (called the *exponent*).

What is the process to follow to write numbers in scientific notation?

Express the following quantities in scientific notation:

- Diameter of the earth (12742000 m)
- Radius of an atomic nucleus (0.000000000000001 m)
- Age of the universe in seconds
- Average mass of a blood cell (0.000000000027g)

Notice that on a calculator, the exponent is written after a letter E (upper or lower case). So Avogadro's number would appear as 6.02E23 and the spacing of carbon atoms in graphene (in m) as 1.42E-10.

A variant on scientific notation is *engineering notation*, which uses exponents that are multiples of three, and significands between 1 and 1000 (so Avogadro's number would be $602 \times 10^{21}$ , and the spacing of carbon atoms would be $142 \times 10^{-12}$ m)

## SI prefixes and "illions"

As well as using scientific notation, we can also add prefixes to units so that they are a convenient size for our problem. For example, buildings designs are usually drawn in millimetres, because that is the precision used in most modern building (and lengths will be whole numbers). But kilometres are much more practical when planning a driving trip.

For large numbers, it can also be convenient to talk in terms of millions, billions, and so on. Expensive houses are costed in millions of dollars (rather than megadollars[2]), and national economic measures like GDPs are typically measured in billions (or trillions) of dollars rather than using SI prefixes.

Fill in the blanks on the missing information in this table:

| SI prefixes | | |
|---|---|---|
| prefix | symbol | Power of 10 |
| peta- | | |
| | | $10^{12}$ |
| | G | |
| | | $10^{6}$ |
| | | $10^{3}$ |
| | h | |
| deca- | | |
| | | $10^{0}$ |
| | d | |
| | | $10^{-2}$ |
| | | $10^{-3}$ |
| micro- | | |
| | n | |
| | | $10^{-12}$ |
| femto- | | |

| "Illions"[3] | |
|---|---|
| name | value |
| Million | |
| | $10^{9}$ |
| Trillion | |
| | $10^{15}$ |
| Quintillion | |

With m, km, kg and yr, it's more common to see million, billon, etc. than mega-, giga-, etc.

## Orders of magnitude

When values vary across a broad scale, it is helpful to think in terms of the exponent, which we call an *order of magnitude*. This idea is closely related to the *logarithm*[4], because

$$log_{10}(10^{m}) = m$$

The concept is particular help when we want to separate the scales of values. For example, the age of the universe is about 13.8 billion years ($1.38 x 10^{10}$ years), while people typically live around 70-80 years, so approximately $10^2$ years. So the universe is some 8 orders of magnitude (10-2) older than the oldest humans.

This difference in orders of magnitude is *independent of the units* we use. This is because the conversion multiples both numbers by the same factor, so their relative difference (the ratio of their values) remains unchanged.

[2] Although it might still cost someone megabucks!

[3] There is also a different definition of billion, trillion, etc. which follows a different pattern. It is an old British system that is rarely seen or used these days

[4] We will look at the logarithm in more detail, as an important scientific function, in Week 5.

## Putting it all together

More practice converting between units, using scientific notation and SI prefixes:

What is the height of the Eiffel tower in microns?

What is the age of the universe is hours?

What is the speed of light in parsecs/hour?

How long would it take to burn off the calories from a Mars Bar

a) running, if I burn 80 kJ/min; or
b) swimming, if I burn 60 kJ/min?

## Switching between grams and moles

Amounts of a substance in chemistry are alternatively measured in grams (ie weight) or moles (ie amount of molecules). To convert between the two, we use the atomic weight of a molecule. For example, water has atomic weight 18.02 g/mol (1 g/mol for each hydrogen atom, and 16 g/mol for the oxygen atom), so one mol of water weights 18.02 g.

Being able to switch between grams and moles is an essential skill in experimental chemistry.

We need a 0.1 mol/l solution of 1-octanol (130.23 g/mol). How many grams of 1-octanol will there be in 1 $cm^3$ of this solution?

A patient has a serum potassium level of 4 mmol/l

(a) How many mmol are present in a 20 ml sample?

(b) How many mg are present in the 20 ml sample?

Apomorphine (267.322 g/mol) can be administered to dogs to induce vomiting after ingestion of toxic substances. The maximum recommended dose is 0.04 mg per kg of bodyweight. If apomorphine comes in a 3.6 mmol/l solution, and Monty weighs 10kg, how many ml of solution should the vet inject? *[Have a go now, or wait a few weeks when we look at more challenging problems such as this]*

Another potentially confusing way of reporting concentrations is by using percentages:

- The notation "%w/w" means "percentage weight of the total weight", so a 1%w/w solution of ethanol in water means that, for every 100 g of *solution*, there is 1 g of ethanol (and so 99 g of water). So "%w/w" can be interpreted as "number of grams of solute per 100 grams of solution."
- The notation "%w/v" means "percentage weight of the total volume". Now, on the face of it, this doesn't make a lot of sense, because here we're not comparing 'like' quantities (weights with weights or volumes with volumes), but quantities that measure different things (a weight versus a volume). This means we have to fix the units of weight and volume we use, and this is based around the fact that water (a very common solvent) weights 1 g per ml. So "%w/v" can be interpreted as "number of grams of solute per 100 ml of solution."

Examples:

A mouthwash contains 0.1% w/v chlorhexidine gluconate. How much chlorhexidine gluconate is contained in 250ml of the mouthwash?

What weight of miconazole is required to make 40g of a cream containing 2% w/w of the drug?

Getting these calculations wrong can sometimes lead to life-and-death situations, so it is important to be able to calculate these reliably (by hand or machine). Incidents where miscalculation led to toxic doses of pharmaceuticals do occur, and there is a famous case of an aeroplane (the Gimli Glider) that ran out of fuel when engineers used the wrong units!