# Quantitative Reasoning

*Is college worth the cost? Should I worry about arsenic in my rice? Can we recycle pollution?* Real questions of personal finance, public health, and social policy require sober, data-driven analyses. This unique text provides students with the tools of quantitative reasoning to answer such questions. The text models how to clarify the question, recognize and avoid bias, isolate relevant factors, gather data, and construct numerical analyses for interpretation. Themes and techniques are repeated across chapters, with a progression in mathematical sophistication over the course of the book, which helps the student get comfortable with the process of thinking in numbers. This textbook includes references to source materials and suggested further reading, making it user-friendly for motivated undergraduate students. The many detailed problems and worked solutions in the text and extensive appendices help the reader learn mathematical areas such as algebra, functions, graphs, and probability. End-of-chapter problem material provides practice for students, and suggested projects are provided with each chapter. A solutions manual is available online for instructors.

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# Quantitative Reasoning
## Thinking in Numbers

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# Preface

This book serves a course designed to teach first-year college students to try to answer topical, real-life questions using numerical analysis, or *quantitative reasoning*. The goal is to develop a student's ability to make carefully reasoned, quantitative arguments utilizing basic mathematical skills. Quantitative reasoning is universally important in all scholarship, and to drive this point home, we have chosen motivating questions from a variety of real-world concerns and a host of academic disciplines. These arguments form the core of the book and are buttressed by skills developed in the appendices. We hope that the topics we explore along the way are both stimulating and fun for the reader.

In many cases, a reliable answer to the questions we pose would require a major effort by an academic researcher. Our aim is not perfection[1] but to familiarize ourselves with the methods and practices of structured, numerical arguments. The pieces of these arguments are usually "simple" aspects of mathematics or statistics. We assume a familiarity but not a facility with these skills. The appendices offer brief reviews of the essential concepts and hints to their use, but the real focus is the application of these skills to answering the questions posed. The pedagogy comes through the extensive, worked examples that compose the chapters. Much of the discussion concerns the tasks of creating a well-posed question, gathering reliable research, constructing a model, and recognizing its underlying assumptions and limitations.

Many themes and techniques are repeated in multiple chapters, although there is a general progression in mathematical sophistication over the course of the book as the topics become more scientific. We view scientific exploration as no different from any field of inquiry and make no attempt to separate it out. The book adopts a narrative style, an intentional contrast from the math-first approach of many curricula.

Answering a variety of questions in essentially the same way should help the student get comfortable with the process of "thinking in numbers." While individual mathematical skills may be practiced and mastered, it is the *coherent assembly* of

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[1] Indeed, our use of economics and social science is pretty shoddy by academic standards. Real-world problems have too many variables for a tidy and rigorous treatment aimed at a novice. Our hope in addressing topical albeit messy questions is that even a somewhat simplified analysis of an actual question will be illuminating and educational.

these skills that quantitative reasoning demands. Professors often expect this ability in their students, and aspects of quantitative reasoning are treated in economics, statistics, and mathematics courses. But few university or high school curricula have courses dedicated to this purpose: this book is a *one-stop shop* for the reader to review or learn basic mathematical skills while learning how to apply them to multistep arguments in a quantitative way. The appendices serve to remind and instruct the reader in the necessary mathematical areas, such as algebra, functions, graphs, probability, and statistics. Readers should be careful to read mathematics slowly. Glossing over the numbers in favor of the text will render the book useless.

### Teacher's Manual

The book is written for a quarter or semester course.

*Syllabus.* For a ten-week quarter course, an instructor could first review the introduction, then proceed by covering one chapter per week, together with a corresponding appendix section, usually with the same number (e.g., Week 1: Chapter 1, Appendix 1). Each of Appendices 7 and 8 can be spread out over two weeks, and Appendix 9 questions may be assigned at any time. The match-up of material won't be perfect, but overlap and review are beneficial to the students. Most of the appendices will be review for most of the students, anyway.

For a semester course, the instructor can spend a full week on the introduction, then do one chapter each week along the lines above, but with two weeks each on Chapters 9 and 10.

*Modularity.* An important feature of the design is that the chapters are independent enough so that an instructor can skip some along the way, if time is running short. This has the benefit of allowing an instructor to delve into any aspect of a chapter without worrying about paying service to a rigid curriculum. If I need to cull material, I like to poll my class to see which chapters interest them most, then adjust the schedule accordingly.

*Homework.* Homework should be assigned, collected, and graded each week: students cannot learn quantitative skills without practice and assessment. A typical homework assignment could be all of (or, more realistically, most of) the exercises from a chapter and a corresponding appendix section, usually with the same number.

It is recommended that the instructor also ask the students to "pre-read" the week's chapter for general concepts (not for specific quantitative skills), then begin the week with a short quiz, 3–5 minutes, testing only rough comprehension: anyone who has done the reading should do well on the quiz.

*Projects.* Students should be assigned projects over the course of the class. Project suggestions are listed at the end of each chapter, though students may have their own ideas. A project report will look roughly like one of the chapters in the text, though not as pedagogical. In a ten-week course, students might be expected to

complete two projects (typically as a group), while a semester-long course might assign three. In either case, the first project should be on the lighter side, as the students will be new to both qualitative and quantitative aspects of reasoning early in the term. I sometimes ask for just an outline for the first project. Instructors and/or teaching assistants should meet with individuals or groups to guide them in beginning their projects, then several times along the way to help ensure they reach some form of conclusion. Try to allot time for in-class presentations of final projects – the students really enjoy it.

Teachers may direct students to the following (as they won't be reading this far into a preface).

### To the Student
Here's all you need to know:

- READ WITH A PEN AND PAPER! You can't read anything with numbers unless you're constantly jotting them down, verifying that you agree with what is written, and deriving results for yourself.
- Go slowly! Read a bit; think a lot. The writing is dense. Points are made, but not repeated. There is no "busy reading." The price tag for this is that you need to consider what is being discussed until it is clear to you.
- The math is in the back. If you're reading something and the math of it is getting too tricky, go the relevant appendix and bone up on your skills. A footnote at the start of each chapter directs you to the most relevant appendix.
- Answer the questions embedded in the text; do the exercises at the end of chapters. The course has homework. Do it. You won't learn anything unless you actually work through examples. The conclusions of the chapters – yes, the sky *is* dark at night! – are meaningless if you haven't developed the ability to explain how any of them are formed.
- "Leave it on the field." One chapter might be difficult. Keep going, and don't let it get to you. Lots of the material will show up again in a later chapter. Over time, you will gain a facility for the techniques involved in the course. ("Leave it on the field" can also mean "try hard in the moment." I assume you will!)

The words above are specific to this course, but as we emphasize the universal nature of reasoning, we should also note that there are universals, best practices, for attending *any* college course:

- Go to class. That is where instructors communicate their deep understanding of the subject to you. Take notes!
- Go to office hours. First, try hard on your own. Then, unless you are sailing through easily, you are likely to need help, and direct contact (not e-mail) with your instructors and teaching assistants is the best way to get it.
- Self-assess. Be frank with yourself about what you do and don't understand (your graded assignments are a good guide). Try to correct what you don't, even if it

means creating extra work for yourself beyond the assignments. Most instructors expect you to do this on your own but may not say so explicitly.

- Talk to classmates and fellow students, but know that you have only understood something when you can do it without guidance.
- Stay confident! Struggle is natural. Meaningful accomplishments come through struggle. Do not lose confidence just because the going gets tough.

Finally, a comment to the reader: the book will try to speak to you in a familiar tone, but will inevitably fail. We all come to life from different cultures and backgrounds and with different experiences and perspectives. A writing passage might sound like a "fresh voice" to some but ring an offensive tone to others. Some may bristle at certain subject matters or turns of phrase. I take responsibility for any offense and apologize. At the same time, I ask readers, if this happens, please do not let *my* shortcomings stand in the way of *your* education!

# Acknowledgments

# Introduction

## The Case for Quantitative Reasoning

> It's the middle of June and you're running late to meet your dad, so you dash in to the national-brand drug store to see if you can find a last-minute Father's Day gift. A display for a wet-dry razor catches your eye: it's on sale for $44.99. The price is a bit more than you wanted to spend, but it's the right thing at the right time.
>
> Your dad loves it.

Lucky for you, eh? Well, actually, there was a lot more to it.

In fact, something extraordinary just happened. Behind the scenes in this scenario were people in the corporate offices analyzing purchasing habits of millions of customers. They tabulate the effect of pricing on the chance of making a sale, as well as placement of the product on the shelves, type of display, etc. They also look at patterns of shoppers near the holidays to determine which items to promote, and how. What was to you a fortuitous find was to them a carefully planned, successful promotion. While a "mom and pop" store can take chances with their promotions and win some, lose some, a national chain with thousands of stores has to invest large sums of money in preparing a promotion, so it can't afford to go by gut and whimsy. Numerical analysis of consumer data can be both crucial to profits and beneficial to the consumer.

Quantitative reasoning helps in more than just sales. It can address everyday questions of home finances or personal politics. In academics, some fields are *built* for a quantitative approach: economics, statistics, the physical sciences, psychology, social sciences, environmental studies. Other fields, such as history (see Chapter 2) and art, have more surprising connections. Students of every stripe – *people in general* – have much to gain from being able to think with numbers.

## Constructing Analytical Models

Understanding human behavior, the environment, and science and technology in the age of Big Data demands quantitative, analytical reasoning. The modern university must shape its curriculum to meet this demand. High school students are trained in mathematics but rarely in the process of assembling mathematical arguments

to address real-life questions from different disciplines: personal finance (*Can I afford this house?*), public health (*How does not washing your hands affect your risk of catching cold?*), or social justice (*Does the pattern of employment reveal discrimination?*). Even whimsical questions (*Can I hear a pin drop?*) can lead to serious scholarly inquiry.

In this book, we address this need by exploring several topical questions in depth, the aim being to teach the reader how to approach questions in a reasoned, quantitative way. We don't just respond with our gut. Surely, emotions play a large role in decision-making. We are humans: we act impulsively, react instinctively; we indulge a good feeling and we hold grudges. The role of emotions must be acknowledged and addressed — even incorporated as a decision-making factor — but *after* all this, we must strive for a dispassionate analysis. The analyses we build are constructed after interpreting the questions in a suitable way. We must understand what the question means to us before attempting to build a response. Each chapter explores one question, crafting a quantitative argument through a comfortable narrative, gathering data, and developing an analytical model along the way. Conclusions are drawn afterward.

Here is a rough template for addressing one of our questions.

1 Frame the Question

  a What are your gut biases/opinions? You'll have to separate these from the analysis.

  b Is the question clear? Does it need to be more specific? Revise/interpret as necessary.

  c What will an answer to this question look like?

2 Build the Model

  a What factors will you involve in your analysis, and why? Why have you omitted others?

  b What numerical data will you need to gather? What sources are you using?

  c What assumptions will you make to be able to model the problem?[2]

  d Having isolated relevant factors, how will you construct your model?

  e What are the sources of errors? How large are they? Be as quantitative as possible.

  f Explain your calculation and produce your result.

3 Analyze Results

  a Check your work!

  b With what degree of certainty can you assert your answer? Be as quantitative as possible.

  c Analyze your result. What does it tell you?

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[2] Most problems will require simplifying assumptions, though some may have an exact answer.

d  Reality check: is the result reasonable, or does it hint at errors? (If the latter, go to Step 2f.)

e  Write a summary.

Each question is unique, but the broad patterns of analysis are well represented in the above blueprint. Even initially simple questions can be addressed in this way – and when we consider them, complexities emerge.

As a basic example, we may ask, *Which costs less, driving to work or taking the train?* If the grumpy conductor terrifies you (1a), you migh interpret "cost" to include emotional torment – which is okay, if you're up front about it. So maybe you're biased to taking the car, but if you decided to measure cost in *dollars,* your bank account will only know the bottom line. We need to make clear (1b) that we mean, say, economic cost. That settled, we then ask (1c), what could an answer look like? Are we looking for a simple car/train recommendation? Do we want to know the amount saved? Or are we looking for a more refined answer: if you're driving such and such a model, then your cost is whatever. Or maybe a probabilistic response: you are this percent likely to save this much money on the train. Okay, for simplicity, suppose we just want a simple car/train recommendation.

Next (2a), we brainstorm the factors that could contribute: cost of transportation, certainly, but also there is the cost of time. The train may take longer, but perhaps you can use it productively by doing work in your seat. Other factors could be the cost of wear and tear on the vehicle or the

> **Sidebar to End All Sidebars**
>
> Sidebars give you short recipes for doing anything. But real problems are complex, and cookie-cutter approaches may not apply. Examples:
>
> • A word problem: you have two water buckets at a well, an eight-gallon and a thirteen-gallon. You want exactly one gallon. What to do? This problem is tough not because it's hard to isolate the relevant information – it's just a hard problem!
>
> • Police police police police police. You see, the "police police" are those who patrol police. Who polices them? Police police police. So the sentence says that this last group is policed by police. Sometimes the complexity is deep and you simply need to slog through.
>
> • Taxation. You want to analyze the result of a tax hike, but there are too many unknowns and variables and not enough data – and if you make a simplifying assumption, you oversimplify so as to become unhelpful. This happens a lot to us in this course, but if you are a politician or professional analyst, you must ultimately decide.
>
> These bullet points are meant to convince you that you cannot reduce all concepts to bullet points. If they didn't, then maybe I just proved my point!

cost of dry cleaning from the train if you always seem to get coffee and calzones spilled on you. We might also consider societal costs: the environmental impacts of the two choices are different, and this can translate in different costs to society, which, ultimately, citizens like you will have to absorb. And so on. To continue the analysis (2b), we collect the facts and figures of transportation, time, wear and tear. To go further, we assume (2c) that we have identified all the factors or that the other factors are negligible by comparison, and we assume, or argue why, our valuations of the various costs are correct. With these factors and assumptions, we will construct our model (2d) by adding up the costs of the train and comparing them to the costs of driving.

After explaining errors in our cost estimation (2e), we move on to adding costs up (2f) and then (3a) checking our calculations. Suppose we determine that driving is cheaper because we need to be at the office to work and we earn money only for the time punched on the clock, and that factor dominated the transportation and other costs. The certainty (3b) of this conclusion may be limited by the fact that we can't determine the driving time with pinpoint accuracy, and if we are quite sophisticated or have good access to data, we may even be able to estimate with some probability what the range of drive times is likely to be and how these might affect (or might not) our recommendation. Now we state the results of our model – driving is cheaper, overall – and interpret it as a recommendation to drive to work today. A reality check (3d) for this recommendation might be the realization that recently there has been a construction project doubling your drive time. Or maybe your daughter had the sniffles last night and there is a good chance that your spouse might need to pick her up from school; the expense of a cab or ride share can change the equation for *today*. If you discover overlooked factors, you will have to re-estimate your costs and run your model again. Once you are sure there are no red flags, you conclude that the recommendation to drive is sound, and you (3e) write a summary of the analysis.

The chapters generally follow the template above, though not formally. Instead, the text flows more freely, breaking along the way to review the mathematical components of our argument. The topical examples from a wide scope of fields should be of interest in their own right but also demonstrate a great range of analytical methods and techniques in action — guiding the reader toward mastery of quantitative reasoning.